South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 3 (2025), pp. 153-168

DOI: 10.56827/SEAJMMS.2025.2103.10 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

PARA-KENMOTSU MANIFOLDS ADMITTING QUARTER-SYMMETRIC METRIC CONNECTION

S. K. Mishra, Giteshwari Pandey*, S. K. Pandey and R. N. Singh

Department of Mathematical Sciences,
A.P.S. University,
Rewa - 486003, Madhya Pradesh, INDIA

E-mail : maths.sandeepmishra@gmail.com, shravan.math@gmail.com,
rnsinghmp@gmail.com

*Department of Mathematics and Computer Science,
Govt. Model Science College,
Rewa - 486001, Madhya Pradesh, INDIA

E-mail : math.giteshwari@gmail.com

(**Received:** *Sep. 10, 2025* **Accepted:** *Dec. 26, 2025* **Published:** *Dec. 30, 2025)*

Abstract: In this article, we examine para - Kenmotsu manifolds equipped with a quarter-symmetric metric connection, focusing on various geometric and curvature properties and we construct an example of a 3-dimensional para-Kenmotsu manifold which confirms the specified metric and structure fulfill the para-Kenmotsu curvature conditions. A unique relation between the curvature tensors of Para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection and the Levi-Civita connection has been established. We explore the characteristics of the locally ϕ-symmetric para-Kenmotsu manifold with respect to the quarter-symmetric metric connection and show that a para-Kenmotsu manifold admitting the quarter-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-symmetric if and only if it is so with respect to the Levi-Civita connection. In addition, we studied ϕ-recurrent para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection and proved that if a para-Kenmotsu manifold is ϕ-recurrent with respect to the quarter-symmetric metric connection, then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.

Keywords and Phrases: Para-Kenmotsu manifolds, quarter-symmetric metric connection, ϕ-recurrent para-Kenmotsu manifolds, ϕ-symmetric para-Kenmotsu manifolds, Einstein manifold.

2020 Mathematics Subject Classification: 53C25, 53C15.

1. Introduction

I. Sato [15] introduced the concept of almost paracontact manifolds as the odd-dimensional counterpart of para-complex structures. This idea has attracted significant interest in differential geometry due to its geometric properties and applications in mathematical physics. While almost contact manifolds [9] are always of odd dimension, almost paracontact manifolds can exist in even dimensions as well. Later, T. Adati and K. Matsumoto [1] studied P-Sasakian and SP-Sasakian manifolds, which represent a specific subclass of almost paracontact Riemannian manifolds. Before Sato's work, Takahashi [18] introduced the concept of almost contact manifolds, including Sasakian manifolds, in the context of a pseudo-Riemannian metric. In 1985, Kaneyuki and Williams [8] formalized the notion of almost paracontact structures on pseudo-Riemannian manifolds of dimension $n = 2m + 1$. Subsequently, Zamkovoy [22] established that every almost paracontact structure can be associated with a pseudo-Riemannian metric of signature $(n+1, n)$. Welyczko [21] introduced the para-Kenmotsu manifold as an analogue of the Kenmotsu manifold [9] within the framework of paracontact geometry. Additionally, significant contributions to the study of para-Kenmotsu manifolds have been made by various researchers ([3], [12], [23]).

The analysis of Riemannian and semi-Riemannian manifolds equipped with different types of connections is essential for understanding their geometric and algebraic structures. In 1975, Golab [6] first introduced the concept of a quarter-symmetric metric connection in differentiable manifolds, extending the idea of semi-symmetric connections.

A linear connection $\tilde{\nabla}$ on a Riemannian manifold (M^n, g) is called a quarter-symmetric metric connection if its torsion tensor $\tilde{T}$ satisfies the relation

$$\tilde{T}(X_1, X_2) = \tilde{\nabla}_{X_1} X_2 - \tilde{\nabla}_{X_2} X_1 - [X_1, X_2], \tag{1.1}$$

which simplifies to

$$\tilde{T}(X_1, X_2) = \eta(X_2)\phi X_1 - \eta(X_1)\phi X_2, \tag{1.2}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. The connection $\tilde{\nabla}$ is a quarter-symmetric metric connection if it satisfies the condition

$$(\tilde{\nabla}_{X_1} g)(X_2, X_3) = 0, \tag{1.3}$$

for all $X_1, X_2, X_3 \in \chi(M)$. When $\phi X_1 = X_1$ and $\phi X_2 = X_2$, the quarter-symmetric connection reduces to a semi-symmetric connection [5]. Rastogi further explored the properties of quarter-symmetric metric connections in his studies ([13], [14]). Additional research on these connections in Hermitian and Kaehlerian manifolds was carried out by Yano and Imai [20]. Over the years, numerous researchers, including Biswas and De [2], Sular [17], De and Mondal [10], De and De [4], Singh and Pandey [16], Haseeb and Prasad [7], Prasad et al. [11] and others, have contributed to this area.

Motivated by these considerations, in this paper, we first derive an explicit relation between the curvature tensors of the Levi-Civita connection and the quarter-symmetric metric connection on para-Kenmotsu manifolds. Using this relation, we prove that a para-Kenmotsu manifold admitting a quarter-symmetric metric connection is locally ϕ-symmetric if and only if it is locally ϕ-symmetric with respect to the Levi-Civita connection. Further, we show that ϕ-recurrent para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection are necessarily η-Einstein manifolds. An illustrative example is also discussed.

2. Para-Kenmotsu manifolds

The concept of an almost para-contact manifold was first introduced by I. Sato [15]. An n-dimensional differentiable manifold M^n is said to possess an almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a characteristic vector field, and η is a 1-form. These satisfy the following conditions

$$\phi^2 X_1 = X_1 - \eta(X_1)\xi, \tag{2.1}$$

$$\eta \circ \phi = 0, \tag{2.2}$$

$$\phi(\xi) = 0, \tag{2.3}$$

$$\eta(\xi) = 1. \tag{2.4}$$

A differentiable manifold equipped with the almost para-contact structure (ϕ, ξ, η) is termed an almost para-contact Riemannian manifold. Furthermore, if the manifold M^n admits a semi-Riemannian metric g that satisfies

$$\eta(X_1) = g(X_1, \xi) \tag{2.5}$$

and

$$g(\phi X_1, \phi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.6}$$

then the structure (ϕ, ξ, η, g), fulfilling conditions (2.2) to (2.6), is known as an almost para-contact Riemannian structure and the corresponding manifold M^n is referred to as an almost para-contact Riemannian manifold [15].

On a para-Kenmotsu manifold [15], the following geometric relations hold

$$(\nabla_{X_1}\phi)X_2 = g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1, \tag{2.7}$$

$$\nabla_{X_1}\xi = X_1 - \eta(X_1)\xi, \tag{2.8}$$

$$(\nabla_{X_1}\eta)X_2 = g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{2.9}$$

$$\eta(R(X_1, X_2, X_3)) = g(X_1, X_3)\eta(X_2) - g(X_2, X_3)\eta(X_1), \tag{2.10}$$

$$R(X_1, X_2, \xi) = \eta(X_1)X_2 - \eta(X_2)X_1, \tag{2.11}$$

$$R(X_1, \xi, X_2) = -R(\xi, X_1, X_2) = g(X_1, X_2)\xi - \eta(X_2)X_1, \tag{2.12}$$

$$S(\phi X_1, \phi X_2) = -(n-1)g(\phi X_1, \phi X_2), \tag{2.13}$$

$$S(X_1, \xi) = -(n-1)\eta(X_1), \tag{2.14}$$

$$Q\xi = -(n-1)\xi, \tag{2.15}$$

$$r = -n(n-1), \tag{2.16}$$

for any vector fields X_1, X_2, X_3, where Q is the Ricci operator *i.e.*

$$g(QX_1, X_2) = S(X_1, X_2).$$

Definition 2.1. *An n-dimensional para-Kenmotsu manifold M^n is termed an η-Einstein manifold if its Ricci tensor S satisfies the equation*

$$S(X_1, X_2) = ag(X_1, X_2) + b\eta(X_1)\eta(X_2),$$

for all $X_1, X_2 \in \chi(M)$, where a and b are scalars.

Example. Let M^3 be a 3-dimensional differentiable manifold with local coordinates (x_1, x_2, x_3) . Define the structure tensors as follows:

The characteristic vector field

$$\xi = \frac{\partial}{\partial x_3}.$$

The 1-form η

$$\eta = dx_3.$$

The structure tensor ϕ

$$\phi\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2}, \quad \phi\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_1}, \quad \phi\left(\frac{\partial}{\partial x_3}\right) = 0. \tag{2.17}$$

We define a semi-Riemannian metric g on M^3 as:

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 - dx_3 \otimes dx_3. \tag{2.18}$$

The Köszul formula for the Levi-Civita connection ∇ on a semi-Riemannian manifold with metric g is given by:

$$\begin{aligned} 2g(\nabla_{X_1} X_2, X_3) &= X_1 g(X_2, X_3) + X_2 g(X_1, X_3) - X_3 g(X_1, X_2) \\ &+ g([X_1, X_2], X_3) - g([X_2, X_3], X_1) - g([X_1, X_3], X_2). \end{aligned} \tag{2.19}$$

Using the Köszul formula, we compute the Levi-Civita connection as follows:

$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_3}, \quad \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3}, \quad \nabla_{\frac{\partial}{\partial x_3}} \frac{\partial}{\partial x_3} = 0, \tag{2.20}$$

$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} = 0, \quad \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_3} = -\frac{\partial}{\partial x_1}, \quad \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_3} = -\frac{\partial}{\partial x_2}. \tag{2.21}$$

The fundamental conditions of a para-Kenmotsu manifold are verified as follows
the condition

$$\phi^2(X_1) = X_1 - \eta(X_1)\xi$$

holds for all vector fields X_1. The compatibility conditions

$$\eta \circ \phi = 0, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1. \tag{2.22}$$

The metric compatibility condition

$$\eta(X_1) = g(X_1, \xi). \tag{2.23}$$

The fundamental equation of an almost para-contact Riemannian manifold

$$g(\phi X_1, \phi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.24}$$

is satisfied for the given metric.

The Riemann curvature tensor is given by

$$R(X_1, X_2)X_3 = \nabla_{X_1} \nabla_{X_2} X_3 - \nabla_{X_2} \nabla_{X_1} X_3 - \nabla_{[X_1, X_2]} X_3. \tag{2.25}$$

Computing for different values of X_1, X_2, X_3, we obtain the following curvature components

$$R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial x_2}, \quad R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) \frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}, \tag{2.26}$$

$$R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\right) \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1}, \quad R\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2}. \tag{2.27}$$

These satisfy the para-Kenmotsu curvature conditions

$$R(X_1, X_2)\xi = \eta(X_1)X_2 - \eta(X_2)X_1. \tag{2.28}$$

$$R(X_1, \xi)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1. \tag{2.29}$$

The Ricci tensor is given by

$$S(X_1, X_2) = -2g(X_1, X_2) + 2\eta(X_1)\eta(X_2). \tag{2.30}$$

The scalar curvature is

$$r = -n(n-1) = -6. \tag{2.31}$$

By directly calculating the Levi-Civita connection using the Koszul formula and determining the curvature tensor, we have confirmed that the specified metric and structure fulfill the para-Kenmotsu curvature conditions.

3. Curvature Tensor of Para-Kenmotsu Manifolds with respect to Quarter-Symmetric Metric Connection

Let $\tilde{\nabla}$ be a linear connection and ∇ be the Levi-Civita connection, satisfying the relation

$$\tilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + A(X_1, X_2), \tag{3.1}$$

where, A is a tensor field of type $(1,1)$. For $\tilde{\nabla}$ to be quarter-symmetric metric connection in M^n, we have [6]

$$A(X_1, X_2) = \frac{1}{2}[\tilde{T}(X_1, X_2) + \tilde{T}'(X_1, X_2) + \tilde{T}'(X_2, X_1)] \tag{3.2}$$

and

$$g(\tilde{T}'(X_1, X_2), X_3) = g(\tilde{T}(X_3, X_1), X_2). \tag{3.3}$$

In the view of equations (1.1) and (3.3), we have

$$\tilde{T}(X_1, X_2) = \eta(X_1)\phi X_2 - g(\phi X_1, X_2)\xi. \tag{3.4}$$

Now using equations (1.1) and (3.4) in equation (3.2), we get

$$A(X_1, X_2) = \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi. \tag{3.5}$$

Hence in view of equations (1.1) and (3.5), a quarter-symmetric metric connection $\tilde{\nabla}$ on para-Kenmotsu manifold is given by

$$\tilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi. \tag{3.6}$$

Further, a quarter-symmetric connection is called a quarter-symmetric metric connection [6] if

$$(\tilde{\nabla}_{X_1} g)(X_2, X_3) = 0. \tag{3.7}$$

Thus equation (3.6) is the relation between quarter-symmetric metric connection and Levi-Civita connection. The curvature tensor $\tilde{R}$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$\tilde{R}(X_1, X_2)X_3 = \tilde{\nabla}_{X_1}\tilde{\nabla}_{X_2}X_3 - \tilde{\nabla}_{X_2}\tilde{\nabla}_{X_1}X_3 - \tilde{\nabla}_{[X_1,X_2]}X_3. \tag{3.8}$$

In view of equation (3.6) above equation reduces to

$$\begin{aligned}\tilde{R}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 + [(\nabla_{X_1}\eta)(X_3)\phi X_2 - (\nabla_{X_2}\eta)(X_3)\phi X_1] \\ &+ [g(\phi X_1, X_3)(\nabla_{X_2}\xi) - g(\phi X_2, X_3)(\nabla_{X_1}\xi)] + \eta(X_3)[(\nabla_{X_1}\phi)(X_2) \\ &- (\nabla_{X_2}\phi)(X_1)] - [g((\nabla_{X_1}\phi)(X_2), X_3) - g((\nabla_{X_2}\phi)(X_1), X_3)]\xi,\end{aligned} \tag{3.9}$$

where

$$\tilde{R}(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3.$$

Using equations (2.7), (2.8) and (2.9) in equation (3.9), we get

$$\begin{aligned}\tilde{R}(X_1, X_2)X_3 =& R(X_1, X_2)X_3 + g(X_1, X_3)\phi X_2 - g(X_2, X_3)\phi X_1 \\ &+ g(\phi X_1, X_3)X_2 - g(\phi X_2, X_3)X_1,\end{aligned} \tag{3.10}$$

which is relation between curvature tensor of connections ∇ and $\tilde{\nabla}$. From equation (3.10), we have

$$\begin{aligned}{}'\tilde{R}(X_1, X_2, X_3, X_4) &= {}'R(X_1, X_2, X_3, X_4) + [g(X_1, X_3)g(\phi X_2, X_4) \\ &- g(X_2, X_3)g(\phi X_1, X_4)] + [g(\phi X_1, X_3)g(X_2, X_4) - g(\phi X_2, X_3)g(X_1, X_4)],\end{aligned} \tag{3.11}$$

where

$${}'\tilde{R}(X_1, X_2, X_3, X_4) = g(\tilde{R}(X_1, X_2)X_3, X_4)$$

and

$${}'R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4).$$

Putting $X_1 = X_4 = e_i$ in equation (3.11) and taking summation over $i,\ 1 \le i \le n$, we get

$$\tilde{S}(X_2, X_3) = S(X_2, X_3) + (1 - n)g(\phi X_2, X_3), \tag{3.12}$$

where $\tilde{S}$ and S are the Ricci tensor of the connections $\tilde{\nabla}$ and ∇ respectively. Again putting $X_2 = X_3 = e_i$ in equation (3.12) and taking summation over $i,\ 1 \le i \le n$, we get

$$\tilde{r} = r, \tag{3.13}$$

where, $\tilde{r}$ and r are the scalar curvature tensor of the connections $\tilde{\nabla}$ and ∇ respectively.
Now, with the help of equation (2.18), we have semi-Riemannian metric g on M^3 as

$$g\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) = 1, \quad g\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right) = 1, \quad g\left(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3}\right) = -1,$$

and

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0 \quad (i \neq j).$$

Then

$$\eta(X_1) = g(X_1, \xi),$$

and

$$g(\phi X_1, \phi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2),$$

for all vector fields X_1, X_2. Thus (ϕ, ξ, η, g) is an almost para-contact metric structure.
Now, for any vector field X_1,

$$\nabla_{X_1}\xi = X_1 - \eta(X_1)\xi,$$

and

$$(\nabla_{X_1}\phi)X_2 = g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1.$$

Hence $(M^3, \phi, \xi, \eta, g)$ is a para-Kenmotsu manifold.
The quarter-symmetric metric connection $\tilde{\nabla}$ is defined in equation (3.6) as

$$\tilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi.$$

It's non-zero components are

$$\begin{aligned}
\tilde{\nabla}_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial x_3}, \\
\tilde{\nabla}_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} &= -\frac{\partial}{\partial x_3}, \\
\tilde{\nabla}_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3} &= -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \\
\tilde{\nabla}_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_3} &= -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}.
\end{aligned}$$

The curvature tensor of $\tilde{\nabla}$ is defined by

$$\tilde{R}(X_1, X_2)X_3 = \tilde{\nabla}_{X_1}\tilde{\nabla}_{X_2}X_3 - \tilde{\nabla}_{X_2}\tilde{\nabla}_{X_1}X_3 - \tilde{\nabla}_{[X_1,X_2]}X_3.$$

Since

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,$$

we obtain

$$\tilde{R}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial x_1} = -\tilde{\nabla}_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}.$$

Verification of curvature relation using equation (3.10),

$$\begin{aligned}\tilde{R}(X_1, X_2)X_3 = R(X_1, X_2)X_3 + g(X_1, X_3)\phi X_2 - g(X_2, X_3)\phi X_1 \\ + g(\phi X_1, X_3)X_2 - g(\phi X_2, X_3)X_1,\end{aligned}$$

and substituting

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_1},$$

we obtain

$$\tilde{R}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1},$$

which coincides with the direct computation.
Hence, this example explicitly illustrates the curvature properties of a para-Kenmotsu manifold admitting a quarter-symmetric metric connection.

The Ricci tensor with respect to ∇ is

$$S(X_1, X_2) = -2g(X_1, X_2) + 2\eta(X_1)\eta(X_2).$$

For $n = 3$, equation (3.12) becomes

$$\tilde{S}(X_1, X_2) = S(X_1, X_2) - 2g(\phi X_1, X_2).$$

In particular,

$$\tilde{S}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = S\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right), \quad i = 1, 2, 3,$$

since $g(\phi\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}) = 0$.

Finally, the scalar curvature with respect to ∇ is

$$r = -n(n-1) = -6.$$

Using

$$\tilde{r} = \sum_{i=1}^{3} \varepsilon_i \tilde{S}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right), \quad \varepsilon_1 = \varepsilon_2 = 1,\ \varepsilon_3 = -1,$$

we obtain

$$\tilde{r} = r.$$

Hence equation (3.13) is verified.

Therefore, this example explicitly confirms equations (3.10), (3.12) and (3.13) for a para-Kenmotsu manifold admitting a quarter-symmetric metric connection.

4. Locally ϕ-Symmetric Para-Kenmotsu Manifolds with respect to Quarter-Symmetric Metric Connection

A para-Kenmotsu manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_{X_4} R)(X_1, X_2)X_3) = 0 \tag{4.1}$$

for all vector fields X_1, X_2, X_3, X_4 orthogonal to ξ. This notion was introduced by Takahashi for Sasakian manifolds [19].

A para-Kenmotsu manifold M^n is said to be ϕ-symmetric [16] if

$$\phi^2((\nabla_{X_4} R)(X_1, X_2)X_3) = 0 \tag{4.2}$$

for arbitrary vector fields X_1, X_2, X_3, X_4.

Analogous to the definition of locally ϕ-symmetric para-Kenmotsu manifolds with respect to Levi-Civita connection, we define a locally ϕ-symmetric para-Kenmotsu manifold with respect to the quarter-symmetric metric connection by

$$\phi^2((\tilde{\nabla}_{X_4} \tilde{R})(X_1, X_2)X_3) = 0, \tag{4.3}$$

for all vector fields X_1, X_2, X_3, X_4 orthogonal to ξ. In the view of equation (3.6), we have

$$\begin{aligned}(\tilde{\nabla}_{X_4} \tilde{R})(X_1, X_2)X_3 =& (\nabla_{X_4} \tilde{R})(X_1, X_2)X_3 + \eta(\tilde{R}(X_1, X_2)X_3)\phi X_4 \\ &+ g(\phi X_4, \tilde{R}(X_1, X_2)X_3)\xi.\end{aligned} \tag{4.4}$$

Now differentiating equation (3.10) covariantly with respect to X_4, we get

$$\begin{aligned}&(\nabla_{X_4} \tilde{R})(X_1, X_2)X_3 = (\nabla_{X_4} \tilde{R})(X_1, X_2)X_3 + \{g(X_1, X_3)g(X_4, X_2) \\ &- g(X_2, X_3)g(X_1, X_4) + 2g(X_1, X_3)\eta(X_2)\eta(X_4) \\ &- 2g(X_2, X_3)\eta(X_1)\eta(X_4)\}\xi + 2\{\eta(X_1)X_2 - \eta(X_2)X_1\}\eta(X_3)\eta(X_4) \\ &- \{\eta(X_1)X_2 - \eta(X_2)X_1\}g(X_3, X_4) + \{g(X_1, X_4)X_2 \\ &- g(X_2, X_4)X_1\}\eta(X_3) + \{g(X_1, X_3)\eta(X_2) - g(X_2, X_3)\eta(X_1)\}X_4.\end{aligned} \tag{4.5}$$

Now taking the inner product of the equation (3.10) with ξ, we get

$$\begin{aligned}\eta(\tilde{R}(X_1,X_2)X_3) = g(X_1,X_3)\eta(X_2) - g(X_2,X_3)\eta(X_1) + g(\phi X_1,X_3)\eta(X_2)\\ - g(\phi X_2,X_3)\eta(X_1).\end{aligned} \tag{4.6}$$

Also from equation (3.10), we have

$$\begin{aligned}&g(\phi X_4,\tilde{R}(X_1,X_2)X_3) = \{g(X_2,\phi X_4) - g(X_2,X_4) + \eta(X_2)\eta(X_4)\}g(X_1,X_3)\\ &+ \{g(X_1,X_4) - g(X_1,\phi X_4) - \eta(X_1)\eta(X_4)\}g(X_2,X_3)\\ &+ g(\phi X_1,X_3)g(X_2,\phi X_4) - g(\phi X_2,X_3)g(X_1,\phi X_4).\end{aligned} \tag{4.7}$$

By the virtue of equations (4.3), (4.4), (4.5) and (4.6), we get

$$\begin{aligned}&\phi^2((\tilde{\nabla}_{X_4}\tilde{R})(X_1,X_2)X_3) = \phi^2((\nabla_{X_4}R)(X_1,X_2)X_3) + \{g(X_1,X_3)g(X_4,X_2)\\ &- g(X_2,X_3)g(X_1,X_4) + 2g(X_1,X_3)\eta(X_2)\eta(X_4)\\ &- 2g(X_2,X_3)\eta(X_1)\eta(X_4)\}\phi^2\xi + 2\{\eta(X_1)\phi^2X_2 - \eta(X_2)\phi^2X_1\}\eta(X_4)\eta(X_3)\\ &+ \{\eta(X_1)\phi^2X_2 - \eta(X_2)\phi^2X_1\}g(X_4,X_3) + \{g(X_1,X_4)\phi^2X_2\\ &- g(X_2,X_4)\phi^2X_1\}\eta(X_3) + \{g(X_1,X_3)\eta(X_2) - g(X_2,X_3)\eta(X_1)\}\phi^2X_4\\ &+ \{g(X_1,X_3) + g(\phi X_1,X_3)\}\eta(X_2)\phi^2(\phi X_4) - \{g(X_2,X_3)\\ &+ g(\phi X_2,X_3)\}\eta(X_1)\phi^2(\phi X_4) + \{g(X_2,\phi X_4) - g(X_2,X_4)\\ &+ \eta(X_2)\eta(X_4)\}g(X_1,X_3)\phi^2\xi + \{g(X_1,X_4) - g(X_1,\phi X_4)\\ &- \eta(X_1)\eta(X_4)\}g(X_2,X_3)\phi^2\xi + g(\phi X_1,X_3)g(X_2,\phi X_4)\phi^2\xi\\ &- g(\phi X_2,X_3)g(X_1,\phi X_4)\phi^2\xi.\end{aligned} \tag{4.8}$$

Consider X_1, X_2, X_3 , X_4 are the orthogonal to ξ, then equation (4.8) yields

$$\phi^2((\tilde{\nabla}_{X_4}\tilde{R})(X_1,X_2)X_3) = \phi^2((\nabla_{X_4}R)(X_1,X_2)X_3). \tag{4.9}$$

Thus, we can state as follows

Theorem 4.1. *A para-Kenmotsu manifold admitting quarter-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-symmetric iff it is so with respect to Levi-Civita connection.*

5. ϕ-Recurrent Para-Kenmotsu Manifolds with respect to Quarter-Symmetric Metric Connection

An n-dimensional para-Kenmotsu manifold M^n is said to be ϕ-recurrent if there exist a non-zero 1-form A such that

$$\phi^2((\nabla_{X_5}R)(X_1,X_2)X_3) = A(X_5)R(X_1,X_2)X_3. \tag{5.1}$$

If $X_1,\ X_2,\ X_3,\ X_5$ are orthogonal to ξ then the manifold is called locally ϕ-recurrent manifold.
If the 1-form A vanishes, then the manifold is reduced to ϕ-symmetric manifold [16].

An n-dimensional para-Kenmotsu manifold M^n is said to be ϕ-recurrent with respect to quarter-symmetric metric connection if there exist a non-zero 1-form A, such that

$$\phi^2((\tilde{\nabla}_{X_5}\tilde{R})(X_1, X_2)X_3) = A(X_5)\tilde{R}(X_1, X_2)X_3, \tag{5.2}$$

for arbitrary vectors $X_1,\ X_2,\ X_3$ and X_5.

Suppose M^n is ϕ-recurrent with respect to quarter-symmetric metric connection, then in view of equations (2.1) and (5.2), we can write

$$\begin{aligned} &g((\tilde{\nabla}_{X_5}\tilde{R})(X_1, X_2)X_3, X_4) - \eta((\tilde{\nabla}_{X_5}\tilde{R})(X_1, X_2)X_3)\eta(X_4) \\ &\qquad = A(X_5)g(\tilde{R}(X_1, X_2)X_3, X_4). \end{aligned} \tag{5.3}$$

By the virtue of equation (4.3), above equation reduced to

$$\begin{aligned} &g((\nabla_{X_5}\tilde{R})(X_1, X_2)X_3, X_4) + \eta(\tilde{R}(X_1, X_2)X_3)g(X_4, \phi X_5) \\ &\quad - \eta((\nabla_{X_5}\tilde{R})(X_1, X_2)X_3)\eta(X_4) = A(X_5)g(\tilde{R}(X_1, X_2)X_3, X_4), \end{aligned} \tag{5.4}$$

which on using equations (3.11), (4.4) and (4.5) takes the form

$$\begin{aligned} &g((\nabla_{X_5}R)(X_1, X_2)X_3, X_4) + 2g(X_2, X_4)\eta(X_1)\eta(X_3)\eta(X_5) \\ &- 2g(X_1, X_4)\eta(X_2)\eta(X_3)\eta(X_5) + g(X_2, X_4)g(X_5, X_3)\eta(X_1) \\ &- g(X_1, X_4)g(X_5, X_3)\eta(X_2) + g(X_1, X_5)g(X_2, X_4)\eta(X_3) \\ &- g(X_2, X_5)g(X_1, X_4)\eta(X_3) + g(X_1, X_3)g(X_4, X_5)\eta(X_2) \\ &- g(X_2, X_3)g(X_5, X_4)\eta(X_1) + \eta(R(X_1, X_2)X_3)g(X_4, \phi X_5) \\ &+ g(X_1, \phi X_3)g(X_4, \phi X_5)\eta(X_2) - g(X_3, \phi X_2)g(X_4, \phi X_5)\eta(X_1) \\ &- \eta((\nabla_{X_5}R)(X_1, X_2)X_3)\eta(X_4) - g(X_1, X_5)\eta(X_2)\eta(X_3)\eta(X_4) \\ &+ g(X_2, X_5)\eta(X_1)\eta(X_3)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4)\eta(X_5) \\ &+ g(X_2, X_3)\eta(X_1)\eta(X_4)\eta(X_5) = A(X_5)g(R(X_1, X_2)X_3, X_4) \\ &+ A(X_5)g(X_1, X_3)g(\phi X_2, X_4) - A(X_5)g(X_2, X_3)g(\phi X_1, X_4) \\ &+ A(X_5)g(X_2, X_4)g(\phi X_1, X_3) - A(X_5)g(X_1, X_4)g(\phi X_2, X_3). \end{aligned} \tag{5.5}$$

Putting $X_3 = \xi$ in above equation and using equations (2.2), (2.3) and (2.4), we get

$$\begin{aligned} & g((\nabla_{X_5}R)(X_1,X_2)\xi,X_4)+2g(X_2,X_4)\eta(X_1)\eta(X_5)-3g(X_1,X_4)\eta(X_2)\eta(X_5) \\ & +g(X_2,X_4)g(X_5,X_1)-g(X_2,X_5)g(X_1,X_4)+\eta(R(X_1,X_2)\xi)g(X_4,\phi X_5) \\ & -\eta((\nabla_{X_5}R)(X_1,X_2)\xi)\eta(X_4)-\{g(X_1,X_5)\eta(X_2)-g(X_2,X_5)\eta(X_1)\}\eta(X_4) \\ & =A(X_5)g(R(X_1,X_2)\xi,X_4)+g(\phi X_2,X_4)\eta(X_1)A(X_5) \\ & -g(\phi X_1,X_4)\eta(X_2)A(X_5). \end{aligned} \tag{5.6}$$

Now, putting $X_1 = X_4 = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} (\nabla_{X_5}S)(X_2,\xi)+(n-2)g(X_2,X_5)+(3n-2)\eta(X_2)\eta(X_5) \\ -\sum_{i=1}^{n} g((\nabla_{X_5}R)(e_i,X_2)\xi)g(e_i,\xi)=A(X_5)S(X_2,\xi). \end{aligned} \tag{5.7}$$

Let us denote the fourth term of left hand side of equation (5.7) by E. In this case E vanishes. Namely, we have

$$\begin{aligned} g((\nabla_{X_5}R)(e_i,X_2)\xi,\xi)=g((\nabla_{X_5}R)(e_i,X_2)\xi,\xi)-g(R(\nabla_{X_5}e_i,X_2)\xi,\xi) \\ -g(R(e_i,X_2)\xi,\nabla_{X_5}\xi) \end{aligned} \tag{5.8}$$

At a fixed point $p \in M$, we choose a local geodesic coordinate system such that $\nabla_{e_i} = 0$ at p. Hence, the Christoffel symbols vanish at p. Therefore $\nabla_{X_2}e_i = 0$. Since R is skew-symmetric, we have

$$g(R(e_i,\nabla_{X_5}X_2)\xi,\xi)=0. \tag{5.9}$$

Using equation (5.9) in equation (5.8), we get

$$\begin{aligned} g((\nabla_{X_5}R)(e_i,X_2)\xi,\xi)=g((\nabla_{X_5}R)(e_i,X_2)\xi,\xi)-g(R(e_i,X_2)\nabla_{X_5}\xi,\xi) \\ -g(R(e_i,X_2)\xi,\nabla_{X_5}\xi). \end{aligned} \tag{5.10}$$

In view of $g(R(e_i,X_2)\xi,\xi) = -g(R(\xi,\xi)e_i,X_2) = 0$ and $(\nabla_{X_5}g) = 0$, we have

$$g((\nabla_{X_5}R)(e_i,X_2)\xi,\xi)-g(R(e_i,X_2)\xi,\nabla_{X_5}\xi)=0, \tag{5.11}$$

Since R is skew-symmetric, we have

$$g((\nabla_{X_5}R)(e_i,X_2)\xi,\xi)=0. \tag{5.12}$$

Using equation (5.12) in equation (5.7), we have

$$(\nabla_{X_5}S)(X_2,\xi)+(n-2)g(X_2,X_5)+(3n-2)\eta(X_2)\eta(X_5)=A(X_5)S(X_2,\xi). \quad (5.13)$$

Now, we have

$$(\nabla_{X_5}S)(X_2,\xi)=\nabla_{X_5}S(X_2,\xi)-S(\nabla_{X_5}X_2,\xi)-S(X_2,\nabla_{X_5}\xi). \quad (5.14)$$

Using equations (2.9), (2.10) and (2.15) in above equation, we have

$$(\nabla_{X_5}S)(X_2,\xi)=-(n-1)g(X_2,X_5)-S(X_2,X_5). \quad (5.15)$$

Using equation (5.15) in equation (5.13), we get

$$S(X_2,X_5)=-g(X_2,X_5)+(4n-3)\eta(X_2)A(X_5). \quad (5.16)$$

Thus, we can state as follows

Theorem 5.1. *If a para-Kenmotsu manifold is ϕ-recurrent with respect to the quarter-symmetric metric connection, then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

6. Conclusions

The development of para-Kenmotsu manifolds is a natural extension of the study of special semi-Riemannian and para-Complex structures in differential geometry. These manifolds provide insight into the rich interplay between geometry, topology and curvature, especially when dealing with indefinite metrics and non-trivial structure tensors. As the study of para-Kenmotsu manifolds continues to evolve new results and applications continue to emerge, particularly in areas where semi-Riemannian geometry plays a crucial role. In the context of these considerations in this paper, we define the torsion of quarter-symmetric metric connection of the para-Kenmotsu manifold, which seems to be non-trivial. We defined a unique relation between the curvature tensors of the para-Kenmotsu manifold with respect to quarter-symmetric metric connection and Levi-Civita connection. We studied locally ϕ-symmetric para-Kenmotsu manifold and ϕ-recurrent para-Kenmotsu manifold with respect to quarter-symmetric metric connection.

References

[1] Adati T. and Matsumoto K., On conformally recurrent and conformally symmetric P-Sasakian manifolds, TRU Math., 13 (1977), 25-32.

[2] Biswas S. C., De U. C., Quarter-symmetric metric connection in an SP-Sasakian manifold, Commun. Fac. Sci. Univ. Ank. Series, 46 (1997), 49–56.

[3] Blaga A. M., Canonical connections para-Kenmotsu manifolds, Novi Sad J. Math., 45(2) (2015), 131-142.

[4] De U. C. and De K., On three dimensional Kenmotsu manifolds admitting a quarter-symmetric metric connection, Azerbaijan J. Math., 1 (2011), 132–142.

[5] Friedmann A., Schouten J. A., Uber die geometrie der halbsymmetrischen ubertragun, Math. Zeitschr, 21 (1924), 211–233.

[6] Golab S., On semi-symmetric and quarter-symmetric linear connections, Tensor, (N.S.), 29 (1975), 249–254.

[7] Haseeb A. and Prasad R., Certain results on Lorentzian para-Kenmotsu manifolds, Bol. Soc. Paran. Mat., Corpus ID: 197864543, (2021).

[8] Kaneyuki S. and Williams F. L., Almost paracontact and parahodge structures on manifolds, Nagoya Math. J., 99 (1985), 173-187.

[9] Kenmotsu K., A class of almost contact Riemannian manifolds, Tohoku Math. Journal, 24 (1972), 93-103.

[10] Mondal A. K. and De U. C., Some properties of a quarter-symmetric metric connection on a Sasakian manifold, Bull. Math. Anal. Appl., 1(3) (2009), 99–108.

[11] Prasad R., Haseeb A., Sen S. and Bilal M., A study on W_6 and W_8-curvature tensors on $(LPK)_n$-manifold with a quarter-symmetric metric connection, Int. J. Anal. Appl., 23, 114 (2025), 1-13.

[12] Raghuwanshi T., Pandey S. K., Pandey M. K. and Goyal A., On generalized W_2-curvature tensor of para-Kenmotsu manifolds, Filomat, 36(3) (2022), 741-752.

[13] Rastogi S. C., On quarter-symmetric connection, C. R. Acad. Sci. Bulgar, 31 (1978), 811–814.

[14] Rastogi S. C., On quarter-symmetric metric connection, Tensor (N.S.), 44 (1987), 133–141.

[15] Sato I., On a structure similar to almost contact structure, Tensor (N.S.), 30 (1976), 219-244.

[16] Singh R. N. and Pandey S. K., On a Quarter-Symmetric Metric Connection in an LP-Sasakian Manifold, Thai Journal of Mathematics., (12) (2014), 357-371.

[17] Sular S., O$\check{z}$g$\ddot{u}$r C., De U. C., Quarter-symmetric metric connection in a Kenmotsu manifold, S. U. T. J. Math., 44 (2008), 297–308.

[18] Takahashi T., Sasakian manifolds with pseudo-Riemannian metric, Tohoku Math. Journal, 21(2) (1969), 644-653.

[19] Takahashi T., Sasakian ϕ-symmetric spaces, Tohoku Math. J., 29 (1977), 91-113.

[20] Yano K., Imai T., Quarter-symmetric metric connections and their curvature tensors, Tensor, (N.S.), 38 (1982), 13–18.

[21] Welyczko J., Slant curves in 3-dimensional normal almost paracontact manifolds, Mediter J. Math., (2013).

[22] Zamkovoy S., Canonical connections on paracontact manifolds, Ann. Global Anal. Geom., 36(1) (2008), 37-60.

[23] Zamkovoy S., On para-Kenmotsu manifolds, Filomat, 32(14) (2018), 4971-4980.